Pergamon

J. Appl. Maths Mechs, Vol. 57, No. 6, pp. 1015–1021, 1993

0021–8928/93 $24.00+0.00

0021–8928(94)E0007-W

QUASILINEAR CONFLICT-CONTROLLED PROCESSES WITH ADDITIONAL RESTRICTIONS†

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(*Received* 22 *June* 1992)

A class of conflict-controlled processes [1–3] with additional ("phase" type) restrictions on the state of the evader is considered. A similar unrestricted problem was considered in [4]. Unlike [5, 6] the boundary of the "phase" restrictions is not a "death line" for the evader. Sufficient conditions for the solvability of the pursuit and evasion problems are obtained, which complement a range of well-known results [5–10].‡

1. The motion of a conflict-controlled object $z=(z_1,\ldots,z_n)$ in the finite-dimensional space R^ν is described by a system of differential equations of the form

$$\dot{z}_i = A_i z_i + \varphi_i(u_i, \upsilon), \quad z_i(0) = z_i^0 \\ z_i \in R^{n_i}, \quad u_i \in U_i, \quad \upsilon \in V \tag{1.1}$$

Here A_i is a specified square matrix of order n_i, U_i and V are non-empty compact subsets of the spaces R^{m_i} and R^m, respectively, and the function $\varphi_i: U_i \times V \to R^{n_i}$ is continuous in all its variables. Here and henceforth $i=1, 2, \ldots, n$; $j=1, 2, \ldots, r$.

The terminal set M consists of sets M_i each of which can be represented in the form

$$M_i = M_i^1 + M_i^2 \tag{1.2}$$

where M_i^1 is a linear subspace of the space R^{n_i}, and M_i^2 is a compact convex set contained in L_i^1, the orthogonal complement to M_i^1 in R^{n_i}. This conflict-controlled process describes a differential game between a group of pursuers $P_1, \ldots, P_n$ and an evader E.

We shall assume that a linear subspace L of the space R^m is specified, together with a system of the form

$$\dot{y} = Ay + \upsilon, \quad y(0) = y^0, \quad \upsilon \in V \tag{1.3}$$

and the set

$$D = \{y | y \in R^m, \langle p_j, \pi y\rangle \leq \mu_j\} \tag{1.4}$$

where A is a specified square matrix of order m, $y^0 \in D$ is a given vector, $p_1, \ldots, p_r$ are unit vectors, $\pi: R^m \to L$ is the orthogonal projection operator, and $\mu_1, \ldots, \mu_r$ are real numbers such that $\operatorname{Int} D \neq \varnothing$.

†*Prikl. Mat. Mekh.* Vol. 57, No. 6, pp. 61–68, 1993.

‡See also: PETROV N. N., Simple pursuit in the presence of phase restrictions. Leningrad, 1984. Deposited in VINITI 27.3.84, No. 1682-84.

Let $T>0$ be an arbitrary number and let σ be a finite decomposition $0=t_0<t_1<\cdots<t_s<t_{s+1}=T$ of the interval $[0, T]$.

Definition 1. A piecewise-programmed strategy Q for the evader E specified in $[0, T]$ with respect to the decomposition σ is a family of mappings b^e, $e=0, 1, \ldots, s$ each of which maps the quantities

$$(t_e, z_1(t_e),\ldots, z_n(t_e), y(t_e)) \tag{1.5}$$

to a measurable function $\upsilon_e(t)$ defined on $t\in[t_e, t_{e+1})$ and such that $\upsilon_e(t)\in V$, $y(t)\in D$, $t\in[t_e, t_{e+1})$.

Definition 2. A piecewise-programmed counterstrategy Q_i for the player P_i with respect to the decomposition σ is a family of mappings c_i^e, $e=0, 1, \ldots, s$ each of which maps the quantities (1.5) and the control $\upsilon_e(t)$, $t\in[t_e, t_{e+1})$ into the measurable function $u_e^i(t)$ defined for $t\in[t_e, t_{e+1})$ and such that $u_i^e(t)\in U_i, t\in[t_e, t_{e+1})$.

We denote the given game by $\Gamma=\Gamma(z^0, D)$.

Definition 3. We shall say that a capture occurs in the game Γ if a $T>0$ exists, and for any decomposition σ of the interval $[0, T]$ for any strategy Q of player E with respect to the decomposition σ piecewise-programmed counterstrategies Q_i exist for the players P_i with respect to the decompositions σ such that there is an instant $\tau\in[0, T]$ and a number g for which $z_g(\tau)\in M_g$.

Definition 4. We say that capture is avoided in the game Γ if for any $T>0$ a decomposition of σ of the interval $[0, T]$ exists, and a strategy Q for the player E with respect to the decomposition σ such that for all counterstrategies Q_i of the players P_i we have $z_i(t)\notin M_i$, $t\in[0, T]$.

2. We will now describe the pursuit scheme. We will denote by π_i the orthogonal projection from R^{n_i} on to L_i^1.

Condition 1. For the point $z^0=(z_1^0, \ldots, z_0^n)$ such that $\pi_i \exp(tA_i)z_1^0\notin M_1^2$ for $t\geqslant 0$ the following relations hold

$$-\overline{\text{con}}(\pi_i \exp(tA_i)z_i^0 - M_i^2)\cap \pi_i \exp((t-\tau)A_i)\varphi_i(U_i, \upsilon)\neq\varnothing \tag{2.1}$$

for all $0\leqslant\tau\leqslant t<+\infty$, $\upsilon\in V$.

Suppose Condition 1 is satisfied for the point z^0. We consider the functions

$$\begin{aligned}\alpha_i(t, \tau, \upsilon) = \max\{\alpha \mid \alpha\geqslant 0,\ -\alpha(\pi_i\exp(tA_i)z_i^0 - M_i^2)\cap \\ \cap\ \pi_i\exp((t-\tau)A_i)\varphi_i(U_i, \upsilon)\neq\varnothing,\ 0\leqslant\tau\leqslant t<+\infty,\ \upsilon\in V\}\end{aligned} \tag{2.2}$$

Put

$$\Omega(t) = \{\upsilon(\cdot)\mid \upsilon\colon\ [0, t]\to V,\ y(\tau)\in D, \tau\in[0, t]\}$$

Condition 2. A time T_0 exists such that

$$\inf_{\upsilon(\cdot)\in\Omega(T_0)}\ \max_i \int_0^{T_0} \alpha_i(T_0, \tau, \upsilon(\tau))d\tau\geqslant 1$$

Theorem 1. Suppose that the point $z^0=(z_1^0, \ldots, z_n^0)$ is such that Conditions 1 and 2 are satisfied. Then capture occurs in the game Γ no later than the time T_0.

The proof is similar to me proof of the theorem in [7, p. 95].

Condition 3. p, $\|p\|=1$, $\mu\in R^1$ exist such that for the set $D_1=\{y\mid y\in R^m,\ \langle p,\ \pi y\rangle\leqslant\mu\}$ we have $D\subset D_1$.

We put

$$d=\max\{\|\upsilon\|\mid \upsilon\in V\},\quad I(g)=\{1,2,\ldots,n+g\}$$

$$\alpha_{n+1}(t,\tau,\upsilon)=\langle\pi\exp((t-\tau)A)\upsilon,p\rangle$$

Condition 4. Continuous functions $\alpha_i^1(t,\ \upsilon)$, $\beta(t,\ \upsilon)$ and continuous non-negative functions $g_i(t,\tau)$, $g(t,\tau)$ exist such that

$$\alpha_i(t,\tau,\upsilon)=g_i(t,\tau)\alpha_i^1(t,\upsilon),\quad \alpha_{n+l}(t,\tau,\upsilon)=g(t,\tau)\beta(t,\upsilon)$$

Let

$$\alpha_{n+1}^1(t,\upsilon)=\beta(t,\upsilon)+a\mu,\quad f(t)=\int_0^t g(t,\tau)d\tau$$

$$\delta(t)=\min_{\upsilon\in V}\max_{e\in I(1)}\alpha_e^1(t,\upsilon),\quad R(t)=d+\delta(t)-a\mu$$

Condition 5. Constants a, c_1, c_2, c_3 exist such that

1. $a\mu\leqslant 0$, $\|\pi\exp(tA)y^0\|\leqslant c_1$ for all $t\geqslant 0$;
2. for any $t>0$ a measurable set $E(t)\subset[0,\ t]$ exists such that

$$\mu(E(t))\leqslant c_2,\quad \int_{E(t)} g(t,\tau)d\tau\leqslant c_3,\quad \min_i g_i(t,\tau)\geqslant g(t,\tau)\forall\tau\in[0,t]\setminus E(t)$$

3. the function $\delta(t)$ is bounded in $[0,\ +\infty)$ and satisfies one of the following two conditions as $t\to+\infty$:

(a) $f(t)\delta^2(t)\to+\infty$ when $a\mu=0$,

(b) $(f(t)\delta(t)\to+\infty$, when $a\mu<0$.

Theorem 2. Suppose that the point $z^0=(z_1^0,\ \ldots,\ z_n^0)$ satisfies Conditions 1, 3, 4 and 5. Then capture occurs in the game Γ.

Proof. Because $D\subset D_1$, it is sufficient to prove the theorem for the game $\Gamma_1=\Gamma(z^0,\ D_1)$. Assume that the assertion of the theorem is false. Then for any $T>0$ a strategy Q exists for player E (with respect to some decomposition σ) such that for any counterstrategies Q_i of players P_i we have $\pi_i z_i(t)\notin M_i^2$ for all $0\leqslant t\leqslant T$. By Condition 1 and the Filippov–Kasten lemma [11] for any i measurable functions $m_i(\tau)\in M_i^2$, $u_i(\tau)\in U_i$, $0\leqslant\tau\leqslant T$, exist which for any fixed $\tau\in[0,\ T]$ are a solution of the equation

$$-\alpha_i(T,\tau,\upsilon(\tau))(\pi_i\exp(TA_i)z_i^0-m_i(\tau))=\pi_i\exp((T-\tau)A_i)\varphi_i(u_i(\tau),\upsilon(\tau))\tag{2.3}$$

At a time τ we assume the value of the control $u_i(\tau)$ (defining the counterstrategy Q_i) to be equal to the lexicographic minimum of all the points u_i for which equality (2.3) is satisfied.

From Cauchy's formula, (2.3) and Condition 4 we obtain

$$\pi_k z_k(T)=\pi_k\exp(TA_k)z_k^0+\int_0^T\pi_k\exp((T-\tau)A_k)\varphi_k(u_k(\tau),\upsilon(\tau))d\tau=$$

$$=\pi_k\exp(TA_k)z_k^0\left(1-\int_0^T g_k(T,\tau)\alpha_k^1(T,\upsilon(\tau)\right)d\tau+\int_0^T\alpha_k^1(T,\upsilon(\tau))g_k(T,\tau)m_k(\tau)d\tau)\tag{2.4}$$

Since the strategy Q is admissible, $\langle p, \pi y(t)\rangle \leqslant \mu$ for all $t \geqslant 0$. From system (1.3) and Condition 4 it follows that

$$\int_0^t g(t,\tau)\beta(t,\upsilon(\tau))d\tau \leqslant \mu - \langle p, \pi \exp(tA)y^0\rangle = \mu_1(t)$$

Let $T_1(t)$, $T_2(t)$ be two subsets of the interval $[0, t]$, such that

$$T_1(t) = \{\tau \mid \tau \in [0, t], \beta(t, \upsilon(\tau)) < \delta(t) - a\mu\}$$

$$T_2(t) = \{\tau \mid \tau \in [0, t], \beta(t, \upsilon(\tau)) \leqslant \delta(t) - a\mu\}$$

Then

$$(\delta(t) - a\mu)G_2 - dG_1 \leqslant \mu_1(t), G_2 + G_1 = f(t)$$

$$\left(G_{1,2} = \int_{T_{1,2}(t)} g(t,\tau)d\tau\right)$$

From the last two relations it follows that

$$G_1 \geqslant [f(t)(\delta(t) - a\mu) - \mu_1(t)]/R(t) \tag{2.5}$$

We consider the functions

$$h_i(t) = 1 - \int_0^t g_i(t,\tau)\alpha_i^1(t,\tau,\upsilon(\tau))d\tau$$

They are continuous, $h_i(0) = 1$ and

$$\sum_i h_i(T) \leqslant n - \delta(T) \int_{T_1(T)} \min_i g_i(t,\tau)d\tau$$

From Condition 5 and inequality (2.5) we obtain

$$\sum_i h_i(T) \leqslant n + c_3\delta(T) - \delta(T)[f(T)(\delta(T) - a\mu) - \mu_1(T)]/R(T) \tag{2.6}$$

From part 3 of Condition 5 and inequality (2.6) it follows that a time T_0 and the number g exist such that the function h_g vanishes at a time $T = T_0$. Hence we conclude from (2.4) that when $T = T_0$

$$\pi_g z_g(T_0) = \int_0^{T_0} g_g(T_0,\tau)\alpha_g^1(T_0,\upsilon(\tau))m_g(\upsilon(\tau))d\tau \in M_g^2$$

The resulting contradiction proves the theorem.

Remark. Theorem 2 remains valid if part 3 of Condition 5 is replaced by the requirement that the right-hand side of inequality (2.6) vanishes for some $T = T_0$.

3. *Example* 1. The pursuers and evader move according to the equations

$$\dot{x}_i = ax_i + u_i, \quad \|u_i\| \leqslant 1, \quad x_i(0) = x_i^0, \quad x_i \in R^m,$$

$$\dot{y} = ay + \upsilon, \quad \|\upsilon\| \leqslant 1, \quad y(0) = y^0, \quad y \in R^m, \quad a < 0$$

The set M_i consists of those points $\{x_i, y\}$, for which $x_i = y$. The restrictions on the evader's coordinates are

$$D = \{y \mid y \in R^m, \langle p_j, y\rangle \leqslant 0\}$$

Assertion 1 [10]. Let $z_i^0 = x_i^0 - y^0 \neq 0$, $n \geqslant m$, $0 \in \operatorname{Intco}\{z_1^0, \dots, z_n^0, p_1, \dots, p_r\}$. Then there is a capture in game Γ.

Assertion 2 [10]. Let $z_i^0 \neq 0$ and $0 \in \operatorname{Intco}\{z_1^0, \dots, z_n^0, p_1, \dots, p_r\}$. Then capture is avoided in game Γ.

Example 2 (the Pontryagin control example with equal coefficients of friction). The motion of the pursuers and evader is described by the equations

$$\dot{x}_{1i} = x_{2i}, \quad \dot{x}_{2i} = ax_{2i} + u_i, \quad x_{1i}, x_{2i} \in R^m, \quad m \geqslant 2, \quad \|u_i\| \leqslant 1$$

$$\dot{y}_1 = y_2, \quad \dot{y}_2 = ay_2 + \upsilon, \quad y_1, y_2 \in R^m, \quad \|\upsilon\| \leqslant 1, \quad a < 0$$

The set M_i consists of the pairs $\{x_{1i}, y\}$, such that $x_{1i} = y$. Restrictions on the evader's geometrical coordinates y_1 have the form

$$D = \{y_1 \mid y_1 \in R^m, \langle p_j, y_1\rangle \leqslant \mu_j\}$$

We put

$$z_{1i} = x_{1i} - y_1, \quad z_{2i} = x_{2i} - y_2, \quad e(t) = a^{-1}(\exp(at) - 1)$$

$$\xi_i(t, z_i^0) = z_{1i}^0 + e(t) z_{2i}^0$$

Then

$$\alpha_i(t, \tau, \upsilon) = e(t-\tau)\alpha_i^1(\xi_i(t, z_i^0), \upsilon), \quad \alpha_{n+j}(t, \tau, \upsilon) = e(t-\tau), \langle p_j, \upsilon\rangle \alpha_i^1(\xi_i, \upsilon) =$$

$$= \|\xi_i\|^{-2}(\langle \xi_i, \upsilon\rangle + [\langle \xi_i, \upsilon\rangle^2 + \|\xi_i\|^2(1 - \|\upsilon\|^2)]^{1/2})$$

$$g_i(t, \tau) = g(t, \tau) = e(t-\tau), \quad f(t) = \int_0^t e(t-\tau)d\tau, \quad E(t) = \varnothing$$

We put

$$z_i^* = z_{1i}^0 - z_{2i}^0 / a = \lim_{t \to \infty} \xi_i(t, z_i^0)$$

Assertion 3. Let $z_i^* \neq 0$, $0 \in \operatorname{Intco}\{z_1^*, \dots, z_n^*, p_1, \dots, p_r\}$ and $n \geqslant m$. Then there is capture in game Γ.

Examples 1 and 2 are solutions of the "cornered rat" and "lion and man" problems [12] in the given formulation.

4. Let us consider in more detail the conflict-controlled process (1.1)–(1.3) for the case when A_i and A are null square matrices. Then the conflict-controlled process is of the simple motion type with mixed player controls and is described by the system of differential equations

$$\dot{z}_i = \varphi_i(u_i, \upsilon), \quad z_i \in R^{n_i}, \quad u_i \in U_i, \quad \upsilon \in V, \quad z_i(0) = z_i^0 \qquad (4.1)$$

Here U_i and V are non-empty compact subsets of the spaces R^{m_i} and R^m, respectively, and the function $\varphi_i(u_i, \upsilon)$ is continuous in its variables. The terminal set M consist of sets M_i each of which is represented in the form (1.2).

The restrictions on the evader have the form

$$\dot{y} = \upsilon, \quad y \in R^m, \quad \upsilon \in V, \quad y(0) = y^0$$
$$D = \{y | \ y \in R^m, \langle p_j, \pi y \rangle \leqslant \mu_j\} \tag{4.2}$$

and $\pi : R^m \to L$ is the orthogonal projection operator on to the linear subspace $L \subset R^m$.

We form the multivalued mappings

$$W_i(z_i^0, \upsilon) = -\overline{\text{con}}(\pi_i z_i^0 - M_i^2) \cap \pi_i \varphi_i(U_i, \upsilon)$$

$$\overline{W}_i(z_i^0, \upsilon) = -\overline{\text{con}}(\pi_i z_i^0 - M_i^2) \cap \text{co}\, \pi_i \varphi_i(U_i, \upsilon)$$

Condition 6. The point $z^0 = (z_1^0, \ldots, z_n^0) \in R^\nu \setminus M$ satisfies the relations $W_i(z_i^0, \ \upsilon) \neq \phi$ for all $\upsilon \in V$.

Condition 7. The point $z^0 = (z_1^0, \ldots, z_n^0)$ satisfies the relations $\overline{W}_i(z_i^0 \ \upsilon) \neq \varnothing$ for all $\upsilon \in V$.

We fix a point z^0 that satisfies Conditions 6 (respectively 7) and introduce the functions

$$\alpha_i(\upsilon) = \max\{\alpha | \ \alpha \geqslant 0, -\alpha(\pi_i z_i^0 - M_i^2) \cap \pi_i \varphi_i(U_i, \upsilon) \neq \varnothing \tag{4.3}$$

$$\overline{\alpha}_i(\upsilon) = \max\{\alpha | \ \alpha \geqslant 0, \ -\alpha(\pi_i z_i^0 - M_i^2) \cap \text{co}\, \pi_i \varphi_i(U_i, \upsilon) \neq \varnothing \tag{4.4}$$

$$\alpha_{n+j}(\upsilon) = \overline{\alpha}_{n+j}(\upsilon) = \langle p_j, \pi\upsilon \rangle$$

We put

$$\delta = \inf_{\upsilon} \max_{e \in I(r)} \alpha_e(\upsilon), \quad \delta_1 = \inf_{\upsilon} \max_{e \in I(r)} \overline{\alpha}_e(\upsilon)$$

$$V_1 = \{\upsilon | \ \alpha_i(\upsilon) = 0, \ i = 1, 2, \ldots, n\}$$

Theorem 3. Suppose the point $z^0 = (z_1^0, \ldots, z_n^0)$ satisfies Condition 6, $\delta > 0$ and at least one of the following two conditions holds: (a) $r = 1$, (b) $0 \notin \overline{\text{co}V_1}$, $\text{co}V_1 \subset \text{con}V_1$.

Then there is a capture in the game Γ.

Proof. If Condition (a) of the theorem is satisfied, Conditions 1 and 3–5 of Theorem 2 are satisfied, from which the assertion follows. Suppose Condition (b) of the theorem is satisfied. Then $\max_j \langle p_j, \ \pi\upsilon \rangle > 0$ for all $\upsilon \in \overline{\text{co}V_1}$. Hence by the Bonneblast–Karlin–Shepley theorem [13, p. 33] $\gamma_j \geqslant 0, \ \gamma_1 + \cdots + \gamma_r = 1$ exist such that

$$\min_{\upsilon \in \text{co}\, V_1} \sum_{j=1}^{r} \gamma_j \langle p_j, \pi\upsilon \rangle > 0$$

Putting

$$p = \gamma_1 p_1 + \ldots + \gamma_r p_r, \quad \mu = \gamma_1 \mu_1 + \ldots + \gamma_r \mu_r$$
$$D_1 = \{y | \ y \in R^m, \langle p, \pi y \rangle \leqslant \mu\}$$

we obtain $D \subset D_1$, $\inf_\upsilon \max_{e \in I(1)} \alpha_e(\upsilon) > 0$, where $\alpha_{n+1}(\upsilon) = \langle p, \ \pi\upsilon \rangle$. This proves the theorem.

Theorem 4. Suppose the point $z^0 = (z_1^0, \ldots, z_n^0)$ satisfies Condition 7, $\delta_1 \leqslant 0$, and a vector $\upsilon_0 \in V$, exists such that

$$\delta_1 = \max_{e \in I(r)} \overline{\alpha}_r(\upsilon_0)$$

Then capture is avoided in the game Γ.

The proof of the theorem is similar to that of Theorem 3 in [4].

Example 1 (see the paper cited in the footnote). Let $n=m$, $M_i=\{0\}$, $\varphi_i(u_i, \upsilon)=u_i-\upsilon$, $U_i=V=D_1(0)$. In this case $\alpha_i(\upsilon)=0$ if and only if $\|\upsilon\|=1$ and $\langle z_i^0, \upsilon\rangle \leqslant 0$. If $n \geqslant m$, then one can take the vectors $z_1^0, \ldots, z_m^0$ to be linearly independent and then, if $\delta>0$ Condition (b) of Theorem 3 is satisfied. We find that there is capture in the game Γ if $n \geqslant m$ and

$$0 \in \operatorname{Intco}\{z_1^0, \ldots, z_n^0, p_1, \ldots, p_r\}$$

Example 2 [9]. Let $n_i=m$, $\varphi_i(u_i, \upsilon)=u_i-\upsilon$, $M_i=\{0\}$, $U_i=V=D_1(0)$, where D is a polyhedron. In this case, it follows from Theorems 3 and 4 that if $n \geqslant m$, then there is capture in the game Γ.

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Translated by R.L.Z.